

# Wilson Loop Renormalization Group Flows

**Joseph Polchinski<sup>1</sup>**

*Kavli Institute for Theoretical Physics  
University of California  
Santa Barbara, CA 93106-4030*

**James Sully<sup>2</sup>**

*Department of Physics  
University of California  
Santa Barbara, CA 93106*

## Abstract

The locally BPS Wilson loop and the pure gauge Wilson loop map under AdS/CFT duality to string world-sheet boundaries with standard and alternate quantizations of the world-sheet fields. This implies an RG flow between the two operators, which we verify at weak coupling. Many additional loop operators exist at strong coupling, with a rich pattern of RG flows.

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<sup>1</sup>joep@kitp.ucsb.edu

<sup>2</sup>sully@physics.ucsb.edu

The Wilson loop operator is a key observable in gauge theories. Studies using AdS/CFT duality [1, 2, 3] have largely focused on the locally BPS loop operator, which couples with equal strength to the gauge and scalar fields. In Euclidean signature,

$$W_{\text{BPS}}[C] = \frac{1}{N} \text{Tr} P e^{\oint_C ds (i\dot{x}^\mu A_\mu + \theta^I \Phi^I)}, \quad \theta^2 = \dot{x}^2. \quad (1)$$

This is dual to the theory in the AdS bulk with a string world-sheet bounded by the curve  $(x^\mu(s), \theta^I(s)/|\theta^I(s)|)$  on  $R^4 \times S^5$ . But what of the simple gauge holonomy

$$W[C] = \frac{1}{N} \text{Tr} P e^{i \oint_C ds \dot{x}^\mu A_\mu} ? \quad (2)$$

This is a natural observable in the gauge theory. Does AdS/CFT duality allow us to calculate its correlators at strong coupling? Indeed, in Ref. [4] a simple prescription for the dual is given.

In this paper we develop further the proposal of Ref. [4]. We point out that it implies an operator renormalization group flow, with the ordinary Wilson loop in the UV and the BPS loop in the IR. Further, at strong coupling there is a much larger set of loop operators, with a rich set of RG flows. These operators do not have any simple weak coupling duals.

We note that much recent work on scattering amplitudes deals with lightlike Wilson loops,  $\dot{x}^2 = 0$ , for which the ordinary and BPS loops coincide. However, the loop equations, at least in their usual form [5], take the string out of the space of locally BPS configurations [3, 6]. Making effective use of the loop equations requires a full understanding of the renormalization of loop operators [7], which is one of the motivations for the present work.

We first review and give further support to the prescription of Ref. [4]. Consider the string world-sheet action in Nambu-Goto form,

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h}, \quad h_{ab} = G_{MN} \partial_a X^M \partial_b X^N. \quad (3)$$

The vanishing of surface term in its variation requires that

$$G_{MN} (h^{ab} n_a \partial_b X^M) \delta X^N = 0 \quad (4)$$

on the boundary of the world-sheet. In terms of the  $AdS_5 \times S^5$  coordinates

$$\frac{ds^2}{R^2} = \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} + du^I du^I, \quad u^I u^I = 1, \quad (5)$$

the boundary lies at  $z = 0$ , or perhaps on a regulating surface  $z = \delta$ . For the BPS loop the embedding is fixed at the boundary,

$$Z = \delta, \quad X^\mu(s, 0) = x^\mu(s), \quad U^I(s, 0) = \theta^I(s)/|\theta(s)|, \quad (6)$$

and so the variation (4) vanishes trivially.

For the ordinary Wilson loop, the dual theory is given by a world-sheet with boundary conditions [4]

$$Z = \delta, \quad X^\mu(s, 0) = x^\mu(s), \quad h^{ab} n_a \partial_b U^I(s, 0) = 0, \quad (7)$$

replacing the Dirichlet condition on the angular variables with a Neumann condition.<sup>1</sup> These boundary conditions are conformally covariant,

$$W[C] \rightarrow W[f(C)], \quad (8)$$

and  $SO(6)$  invariant. At weak coupling the gauge holonomy is the unique operator with these properties: conformal covariance implies that it is constructed from the line integral of dimension one operators, and  $SO(6)$  invariance excludes the scalars. As a further symmetry check, we verify in the appendix that the Dirichlet condition is locally supersymmetric, and the Neumann condition is not.

One can give a formal derivation by introducing an independent world-line field  $u^I(s)$  with  $u \cdot u = 1$ , and averaging the loop operator

$$\int \mathcal{D}u \, \text{Tr} \, P e^{\oint_C ds (i\dot{x}^\mu A_\mu + |\dot{x}| u^I \Phi^I)}. \quad (9)$$

On the world-sheet the sum over Dirichlet conditions produces a free boundary condition, giving the Neumann condition as an equation of motion. Expanding the loop operator in powers of  $u \cdot \Phi$ , the linear term averages to zero, the quadratic term averages to a contour integral of  $\Phi^2$ , which is irrelevant, and so on, and only the holonomy survives.

This is somewhat surprising, because it implies that at large 't Hooft coupling  $\lambda$  the expectation values for BPS operators with fixed  $\hat{\theta}^I = \theta^I(s)/|\theta^I(s)|$  are the same as for the pure gauge operator. They are governed by the same saddle points, at constant  $u^I = \hat{\theta}^I$ , and differ only in the determinants which give an effect subleading in  $1/\sqrt{\lambda}$ . Thus the force between a fundamental and an antifundamental does not depend on whether they couple to a common scalar  $\Phi^I \hat{\theta}^I$ .<sup>2</sup>

This is certainly not true at weak coupling, where the scalar exchange is equal in magnitude to the gauge exchange. However, it is consistent with some earlier observations. In Ref. [3] it was noted that the BPS loop satisfies zig-zag symmetry [9] at strong coupling, whereas this is only expected for the simple loop. In Ref. [7] it was noted that the AdS

<sup>1</sup>In Ref. [3] the *supersymmetric* boundary condition appears as a Neumann condition, but in terms of the variable  $Y^I = zU^I$ .

<sup>2</sup>The potentials from Neumann and Dirichlet boundary conditions for have been compared in Refs. [8], including the case of coupling to different scalars.

Wilson loop satisfies the same loop equation as in pure gauge theory, and it was conjectured that this is a universal behavior at large  $\lambda$ .

There is another way to think about these two boundary conditions. Consider a string running radially in  $AdS_5 \times S^5$ , e.g. along  $u^6 = 1$ . Its world-volume is an  $AdS_2$ , and the world-sheet fluctuations  $U^i$  for  $i = 1, \dots, 5$  are described by massless fields in  $AdS_2$ . Near the AdS boundary, one then has

$$U^i \approx \alpha^i z^{\Delta_-} + \beta^i z^{\Delta_+}, \quad \Delta_{\pm} = \frac{d}{2} \pm \sqrt{m^2 + d^2/4} = \frac{1}{2} \pm \frac{1}{2}. \quad (10)$$

The Dirichlet quantization sets  $\alpha^i = 0$ , and the Neumann quantization sets  $\beta^i = 0$ . Thus, these are the standard and alternate quantizations in the sense of Refs. [10]. Near the AdS boundary every world-sheet is asymptotically  $AdS_2$  near any smooth point of the loop  $C$ , so this reasoning applies more generally.

Consequently, the insertion of  $\Phi^i$  into the loop, which should be dual to the boundary perturbation of  $U^i$ , has dimension 0 in the ordinary loop (up to a higher correction [4] to be discussed below) and dimension 1 in the BPS loop. The latter can also be seen from the fact that the insertion of  $\int ds \Phi^i$  is just an infinitesimal  $SO(6)$  rotation of the loop, and so marginal. Using the conformal flatness of the  $AdS_2$  metric  $ds^2 = (d\tau^2 + dz^2)/z^2$  we can immediately write down the bulk-to-bulk propagators

$$\langle U^i(\tau, z) U^j(\tau, z') \rangle = \frac{\delta^{ij}}{2\sqrt{\lambda}} \left( -\ln(|\tau - \tau'|^2 + |z - z'|^2) \pm \ln(|\tau - \tau'|^2 + |z + z'|^2) \right), \quad (11)$$

with the upper sign for the standard quantization and the lower sign for the alternate quantization. Taking the boundary limit  $z, z' \rightarrow 0$  gives the  $\Delta = 1$  two-point function for the standard quantization and a logarithmic two-point function for the alternate quantization. In the latter case  $U^i$  is not a good quantum field due to IR divergences, but its time derivative is. For a closed Neumann Wilson loop  $U^i$  has a normalizable zero mode, whose integral enforces  $SO(6)$  invariance.

This interpretation immediately suggests an interesting RG flow. If instead of one of the pure boundary conditions we impose the mixed condition

$$\alpha^i + \beta^i f = 0, \quad (12)$$

then the  $\alpha$  term will dominate near the boundary and the  $\beta$  term near the horizon: we have an RG flow [11] from the ordinary Wilson loop in the UV to the BPS loop in the IR. A more precise and  $SO(6)$  invariant way to formulate this is to begin with the Neumann theory and add a boundary perturbation

$$-\frac{\sqrt{\lambda}}{2\pi f} \int d\tau U^6(\tau, 0) \quad (13)$$

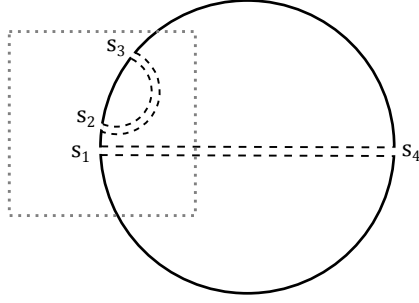


Figure 1: Scalar loop correction to the scalar vertex.

on the world-sheet. Since  $U^6$  has dimension zero in this quantization, this is relevant: it is negligible in the UV and dominant in the IR. At low energy the path integral will be dominated by the configuration of minimum action (13), extending in the  $U^6$  direction. The fluctuations around this configuration satisfy the boundary condition (12).

Thus we identify the perturbed world-sheet with the family of operators

$$W_\zeta[C] = \frac{1}{N} \text{Tr} P e^{\oint_C d\tau (i\dot{x}^\mu A_\mu + \zeta |\dot{x}| \theta^I \Phi^I)}, \quad (14)$$

where here  $\theta^I = \delta^{I6} |\dot{x}|$ . These interpolate between the ordinary loop at  $\zeta = 0$  and the BPS loop at  $\zeta = 1$ . That is,  $\zeta = F(f\mu)$ , increasing from 0 to 1 as  $f\mu$  decreases from  $\infty$  to 0 (the renormalization scale is included for dimensions). Negative values of  $f$  give a world-sheet extended in the opposite direction on  $S^5$ , and correspond to the range  $0 > \zeta > -1$  where  $-1$  is again a BPS loop. Note that the double-trace interpretation [11] of the flow for bulk fields is not relevant here, rather we are inserting additional scalars into the single Wilson trace.

One would expect this flow also to be evident at weak coupling. Expanding perturbatively, power-counting gives a log divergence and so a possible contribution to the running of  $\zeta$  whenever a group of vertices (on the Wilson contour and/or in the volume) approach one end of a scalar propagator attaching to the contour. At order  $g^2 \zeta^3$  the only graph is Fig. 1, as  $s_2, s_3 \rightarrow s_1$ : The vertex correction is

$$\frac{\lambda \zeta^3 |\dot{x}(s_1)|}{8\pi^2} \int_{s_1}^{s_4} ds_2 \int_{s_1}^{s_3} ds_3 \frac{|\dot{x}(s_2) \dot{x}(s_3)|}{(x(s_2) - x(s_3))^2} \quad (15)$$

plus a similar piece from  $s_2, s_3 < s_1$ . One can evaluate this readily in dimensional regularization, but we will take a spatial regulator, requiring  $s_3 - s_2 > \epsilon$ . Defining  $s_\pm = (s_3 - s_1) \pm (s_2 - s_1)$ , the contribution from the region near  $s_1$  is then

$$\frac{\lambda \zeta^3 |\dot{x}(s_1)|}{16\pi^2} \int_\epsilon ds_+ \int_\epsilon^{s_+} \frac{ds_-}{s_-^2} = \frac{\lambda \zeta^3 |\dot{x}(s_1)|}{16\pi^2} \int_\epsilon ds_+ \left( \frac{1}{\epsilon} - \frac{1}{s_+} \right), \quad (16)$$



We have not given a strong-coupling prescription for the loop with  $|\zeta| > 1$ , but even at weak coupling this range is problematic: the flow (17) leads to the BPS loop in the IR, but diverges in the UV and there may be no continuum operator. Note that we could also consider complex  $\zeta$ . Almost all flows still go to the Wilson loop in the UV and the BPS loop in the IR, the exception being pure imaginary  $\zeta$  which diverges in the IR.

The fields  $X^\mu$  have  $m^2 R^2 = 2$  [14], so  $\Delta_+ = 2$  and  $\Delta_- = -1$ . The standard quantization corresponds to the insertion of  $F_{\mu\nu}\dot{x}^\nu$  [7]. The alternate quantization should correspond to the Fourier transform, the momentum loop of Ref. [15], but the negative dimension is unacceptable: apparently the Fourier transform does not exist at strong coupling. One could regulate this by inserting a factor of  $\exp(-m \oint ds (x - x_C(s))^2)$ , producing a loop that flows to the momentum loop in the UV and to a smeared position-space loop in the IR. Other such weighted sums over Wilson loops have been considered in Refs. [16]. They may be useful in disentangling the loop equations.

It is satisfying to have an AdS interpretation for the CFT operator  $W[C]$ , but there is still a mismatch in the other direction. One could satisfy the boundary equation (4) by taking Dirichlet conditions on some of the  $U^I$  and Neumann conditions on others, say

$$U^1 = U^2 = U^3 = 0, \quad \partial_n U^4 = \partial_n U^5 = \partial_n U^6 = 0. \quad (18)$$

(Note that the constraint  $U^I \partial_n U^I$  is satisfied.) This is different from any loop considered above, and the different world-sheet determinants will give a distinct amplitude. Like the Wilson and BPS loops it is conformally covariant, but there is no candidate for a weak-coupling dual.

To get some insight consider perturbing the Neumann theory by the boundary operator

$$\int d\tau \left( \sum_{I=1}^3 U^I(\tau, 0)^2 - \sum_{I=4}^6 U^I(\tau, 0)^2 \right). \quad (19)$$

The argument again has dimension 0 at strong coupling so this is relevant, and over long distances the loop will want to sit on the 2-sphere  $U^1 = U^2 = U^3 = 0$  where the action is minimized; the boundary conditions tangent to this  $S^2$  remain Neumann. Thus, under this perturbation the ordinary Wilson loop flows to the loop (18).

At next order in  $1/\sqrt{\lambda}$ , the operator  $C_J(U)$  with spherical harmonic  $C_J$  has dimension  $J(J+4)/\sqrt{\lambda}$  [4]. One way to see this is to think of  $C_J(U)$  as an open string vertex operator, for which the leading dimension is  $-\alpha' \nabla^2 / R^2$ . This increases with decreasing coupling, and at zero coupling it reaches the dimension  $J$  of the insertion  $C_J(\Phi)$ . Thus, at some coupling the dimension of  $C_2(U)$  passes through 1, and the perturbation (19) switches from

relevant to irrelevant. The flow then reverses, and the Wilson operator is the IR fixed point (the  $SO(3) \times SO(3)$  symmetry prevents the flow from going to the BPS loop). It is not clear whether there is any UV fixed point for this reverse flow — the operator with insertion  $C_2(\Phi)$  is perturbatively nonrenormalizable.

General perturbations  $V(U)$  define a large set of loop operators at strong coupling. All flow to the Neumann loop in the UV, and functions with a unique minimum flow to the Dirichlet loop in the IR. Potentials with continuous degenerate minima flow to other loops as above. Potentials with discrete degenerate minima will flow to a sum over kinked loops. As the coupling is decreased the number of perturbatively renormalizable operators decreases in steps, leaving just the  $\zeta$ -loops (14) at sufficiently weak coupling.

Finally, there should be a parallel story for 't Hooft loops and D-strings.

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## A Supersymmetry

Linearizing around a radial string in  $AdS_5 \times S^5$ , the three bosonic fluctuations in the  $AdS_5$  directions have  $m^2 = 2$  in AdS units, the five bosonic fluctuations along the  $S^5$  are massless, and the eight fermions have  $|m| = 1$  [14]. We write the action keeping only the unbroken  $SO(1,1) \times SO(3) \times SO(5)$  symmetry manifest. In particular the fermion  $\psi$  and supersymmetry transformation  $\epsilon$  are in the Majorana representation  $(2, 2, 4)$ . We define gamma matrices for the respective factors,  $\gamma^a$  for  $a = 0, 4$ ,  $\sigma^i$  for  $i = 1, 2, 3$ , and  $\tau^p$  for  $p = 5, 6, 7, 8, 9$ ; matrices from different sets are mutually commuting. Building on the basic  $AdS_2$  supermultiplet [17, 18], the action and supersymmetry transformation are [19]

$$S = \frac{1}{2} \int d^2x \sqrt{-g} \left( -\partial_a X^i \partial^a X^i - 2X^i X^i - \partial_a X^p \partial^a X^p + i\bar{\psi} \gamma^a D_a \psi - m\bar{\psi} \psi \right) \quad (A.1)$$

and

$$\begin{aligned} \delta X^i &= i\bar{\psi} \gamma_5 \sigma^i \epsilon, \quad \delta X^p = \bar{\psi} \tau^p \epsilon, \\ \delta \psi &= -i\sigma^i (X^i + i\gamma^a \partial_a X^i) \gamma_5 \epsilon - \tau^p (X^p + i\gamma^a \partial_a) \epsilon. \end{aligned} \quad (A.2)$$

We study the supersymmetry of the boundary conditions using the approach of Ref. [20]. Expanding the solutions of the field equation near the boundary gives

$$\begin{aligned} X^i &\sim z^{-1}\alpha^i(t) + z^2\beta^i(t), \\ X^p &\sim \alpha^p(t) + z\beta^p(t), \\ \psi &\sim z^{-1/2}\alpha^\psi(t) + z^{3/2}\beta^\psi(t), \end{aligned} \tag{A.3}$$

where  $\gamma^4\alpha^\psi = i\alpha^\psi$ ,  $\gamma^4\beta^\psi = -i\beta^\psi$ . The supersymmetry parameter is  $\epsilon = z^{-1/2}\eta$  for arbitrary constant spinor  $\eta$ , and the variations take the form

$$\begin{aligned} \delta\alpha^i(t) &= i\bar{\alpha}^\psi\gamma_5\sigma^i\eta, \quad \delta\beta^i(t) = O(\dot{\beta}^\psi), \\ \delta\alpha^p(t) &= O(\dot{\alpha}^\psi), \quad \delta\beta^p(t) = \bar{\beta}^\psi(t)\tau^p\eta, \\ \delta\alpha^\psi(t) &= -\tau^p\alpha^p(t)\eta + O(\dot{\alpha}^i), \quad \delta\beta^\psi(t) = -i\sigma^i\beta^i(t)\gamma_5\eta + O(\dot{\beta}^p). \end{aligned} \tag{A.4}$$

For the massive  $X^i$ , the only allowed quantization is the standard  $\alpha^i = 0$ . The  $SO(3) \times SO(5)$  symmetry implies common boundary conditions on all fermionic components, and supersymmetry then requires  $\alpha^\psi = \alpha^p = 0$ , i.e. the Dirichlet condition for the  $S^5$  variables. The alternate  $\beta^p = 0$  quantization is not supersymmetric.

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